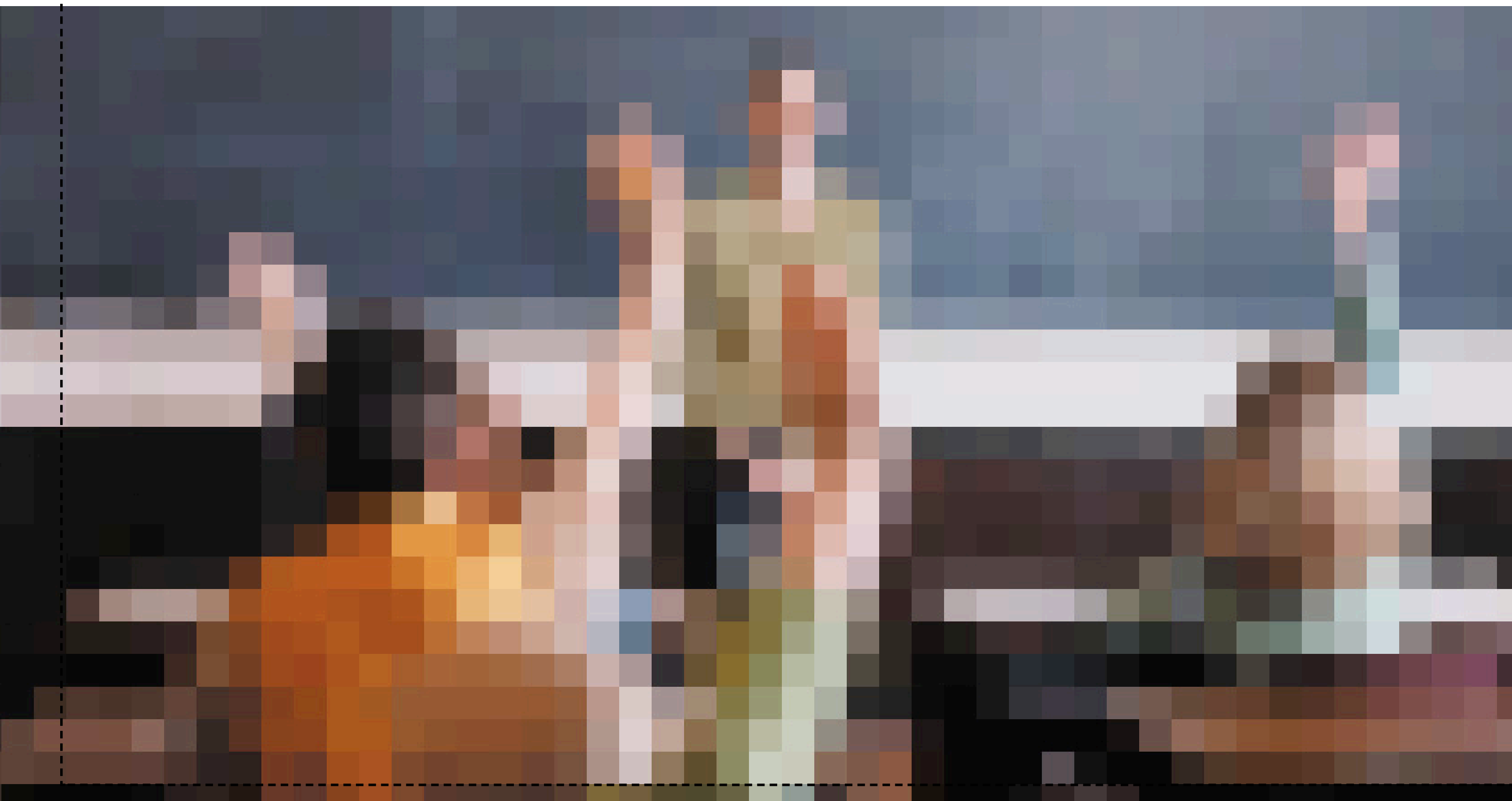
Building Explicit and Recursive Forms of Patterns with the FUNCTION GAME

RHETA N. RUBENSTEIN

THE FUNCTION GAME IS A POWERFUL and motivating tool for engaging middle-grades students in mental mathematics, problem solving, communication, and inductive reasoning (Rubenstein 1996). The game can also be used to help students achieve the goals of NCTM's Algebra Standard for grades 6–8; that is, to "represent, analyze, and generalize a variety of patterns with tables, graphs, words, and, when possible, symbolic rules" (NCTM 2000, p. 222). (For a simple electronic version of the game, use the applet on the CD-ROM in Cuevas and Yeatts [2001].) This article will show how the function game format serves as a launchpad to help students build, distinguish, and translate between two basic forms of patterns.

RHETA RUBENSTEIN, *rhetar@aol.com, teaches mathematics content and methods courses at the University of Michigan—Dearborn. She is interested in making meaningful and interesting mathematics accessible to all students.*

To play the function game, also known as "guess my rule," "the computer game," or "the input/output game," the teacher or a student acts as the "computer." Players—the whole class or a smaller group—offer one input number at a time. The "computer" follows some fixed but secret rule to produce the related "output." The input numbers and associated output results are recorded on a table. For example, for a game in which the rule is



"the output is 3 times the input plus 1," the table of a game in progress might look like the following:

INPUT	OUTPUT
7	22
12	37
10	31
25	76

The object is to guess the rule. To keep the game open, if a player knows the rule, he or she is asked not to state it. Instead, the player may demonstrate his or her knowledge by telling the output for a given input. (If the player is wrong, he or she either does not know the rule or miscalculated. If the player is right, he or she probably knows the rule.) When a number of people know the rule, someone is invited to state it. Sometimes, alternative versions are given. For example, a student may state the rule above as "double the number, add it again, then add 1." Then students are challenged to decide why that rule and "triple the input plus 1" are always equivalent.

One strategy students usually figure out after playing a few games is to use 0 for an input. Another strategy is to give input values that are whole numbers in consecutive order. More astute players who are trying to guess the rule above might produce a table that looks like the following:

INPUT	OUTPUT
0	1
1	4
2	7
3	10

At this point, students usually figure out a pattern in the output column of "add 3 to the previous number." This strategy gives students a way to build chains of consecutive output results but stumps them when they try to determine the output for an input of, say, 30. To find that result, students need a rule that moves directly from input to output. The same problem occurs when presenting students with a sequence and asking them to continue it. For example, students are often able to find the next three terms in the sequence 12, 24, 48, 96 by recognizing that each term is twice the previous term. They have trouble, however, giving a formula for the tenth term or, more abstractly, the *n*th term (e.g., $6 \bullet 2^n$). As one common example, when students are presented with the sequence 1, 4, 9, 16, 25, 36, 49, 64, . . . , rather than seeing the perfect squares jump off the page, many see that they can move from term to term by adding the next odd number. For example:

$$1 + 3 = 4, \quad 4 + 5 = 9, \quad 9 + 7 = 16, \quad 16 + 9 = 25$$

The students' pattern is beautiful, but perfect squares ought to be such "good friends" that students recognize them immediately, especially when a whole group appears together!

These examples show that students look for patterns in at least two distinct ways. The following paragraphs describe how to help students move between these two perspectives.

Functions, Sequences, and Recursive versus Explicit Rules

ALTHOUGH THE FUNCTION GAME CAN BE played with all kinds of input numbers, such as fractions, integers, and so on, when the input consists of whole numbers (0, 1, 2, 3, . . .), the function can be thought of as a sequence. A rule for the sequence can be expressed recursively or explicitly. Consider, for example, the game above with the output 1, 4, 7, 10, ... Using the students' generally intuitive notion of "adding 3," we can build a recursive, or step-by-step, rule:

start = 1
next = current + 3

The first step tells how the sequence begins. It anchors the later steps. The second step tells how to go from one output to the next. The metaphor I share with students is that once I know how to begin, I can just "look over my shoulder" at the previous number to figure out how to find the next number. We often need a conversation at this point to convince students that both parts of the "rule" are necessary. "Why do I need to know the starting number? Suppose the first number were 10 rather than 1. Then what would the following numbers be? Is the sequence the same?" Ultimately, students recognize that the "start" matters.

The "start, next, current" notation is used in *Mathematics in Context* (Encyclopaedia Britannica Educational Corporation 1998). Other publishers use "next = now + 3" or "new = old + 3." In all these textbooks, the notation has been designed to be student friendly, unlike the more abstract and traditional subscripted sequence or function notations, such as the following:

$$t_1 = 1 \qquad f(1) = 1$$
$$t_n = t_{n-1} + 3 \qquad f(x) = f(x-1) + 3$$

The rule that makes this round of the function game easy to play, however, is simply $y = 3x + 1$, or output = 3(input) + 1. This rule is an *explicit* (or

INPUT	OUTPUT	
0	1	Start at 1
1	4	+3
2	7	+3
3	10	+3
x	$y = 3x + 1$	

Recursive Rule ↓

Explicit Rule →

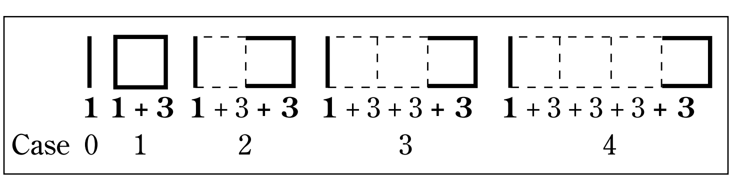
Fig. 1 Distinguishing recursive and explicit rules

NO. OF PENS	NO. OF PANELS (OR TOOTHPICKS)	PATTERN A	
1	4	4	4
2	7	4 + 3 =	4 + 1(3)
3	10	4 + 3 + 3 =	4 + 2(3)
4	13	4 + 3 + 3 + 3 =	4 + 3(3)
5	16	4 + 3 + 3 + 3 + 3 =	4 + 4(3)
n			$4 + (n - 1)3$
30			4 + (29)3 = 91

(a)

NO. OF PENS	NO. OF PANELS (OR TOOTHPICKS)	PATTERN B	
0	1	1	
1	4	1 + 3	1 + 1(3)
2	7	4 + 3 + 3 =	1 + 2(3)
3	10	4 + 3 + 3 +3 =	1 + 3(3)
4	13	4 + 3 + 3 + 3 + 3 =	1 + 4(3)
5	16	4 + 3 + 3 + 3 + 3 + 3 =	1 + 5(3)
n			$1 + n(3)$
30			1 + (30)3 = 91

(b)



(c)

Fig. 2 Using tables to build explicit rules from recursive rules

closed or *direct*) rule. With an explicit rule, one can take any input and find the corresponding output directly.

Students need to be able to distinguish these two ways of thinking about rules, that is, using a step-by-step, or recursive, form versus using a direct, or explicit, form. When students talk about a problem, they need to know if one person is offering an explicit rule while another is using a recursive rule. For example, consider asking students to describe patterns for the even numbers 2, 4, 6, 8, 10, ... One student may say, "add 2" and another may say, "multiply by 2." They appear to be in conflict! Students must realize that "add 2" is part of a recursive rule: "Start at 2 and add 2 to the previous number to get the next." In contrast, "multiply by 2" is an explicit rule: "Multiply the input by 2 to get the output."

After struggling for some time with teaching this distinction, I discovered that the function game table helps tremendously. As shown in **figure 1,** a recursive rule goes from output to output down the right-hand column. An explicit rule goes from input to output across the table. The table clarifies the two types of rules and how they each work. In particular, when students are asked to find rules for sequences presented simply as consecutive terms, they typically do not see the "input," or term, number and have difficulty using this "hidden" number in building a rule. The input/output table makes the input explicit.

Using Game Tables to Build Explicit Rules from Recursive Rules

BECAUSE SO MANY STUDENTS HAVE A NATURAL inclination to look first for recursive rules, finding explicit rules may be hard for them. Again, the function game table can build a bridge between the two forms. Assume that students have been asked to solve the following problem: "A construction crew at an agricultural fair is building square animal pens. How many panels does the crew need to build a line of 30 pens?" The problem may be modeled with toothpicks, as shown in **figure 2.** Note that students are working with the same pattern shown earlier but this time, in a geometric context. They are asked to include a third column (as shown in **figure 2a**) to show how the recursive rule, "add 3 to the previous number," can be used to find an explicit rule. The third column shows the structure of the pattern explicitly. After seeing an example or two, students usually begin to realize that they are just adding more 3s to the starting value. The number of 3s added is one less than the input number because the 3 is not added the first time. Consequently, an explicit formula for the output can be derived: $4 + (n - 1)3$.

For many students, the $(n - 1)$ idea is difficult. For these students, we offer another solution method, shown in **figure 2b.** At this point, ask, "If you can add 3 to move forward, what must you do to move backward?" Students realize that they must subtract 3. With this part of the rule in mind, they move backward one row from the starting row to consider an input of 0. The output for 0 must be $4 - 3 = 1$. As shown in **figure 2c,** the "0 case" can be envisioned geometrically. Imagine just one toothpick at the left of the display. Then, for each successive square, add three toothpicks in the form of a backward C. Many students benefit by seeing geometrically the connection between the numerical pattern and its physical construction. Now we can create a formula in which the number of 3s added is precisely the number of squares, or the input number. The formula is $1 + 3n$.

The particular problem illustrated asks students to find the output for the thirtieth case. With a general formula now in hand, they can substitute and evaluate for $n = 30$ or any other input.

1. Graph the data from the agricultural pen pattern (**fig. 2**).

   Let the x-axis show the number of pens scaled by ones.

   Let the y-axis show the number of panels scaled by fives.

2. What is the shape of the graph?

3. What are its characteristics? Slope? y-intercept?

4. What do the graph's characteristics have to do with the original pattern?

Sample graph:

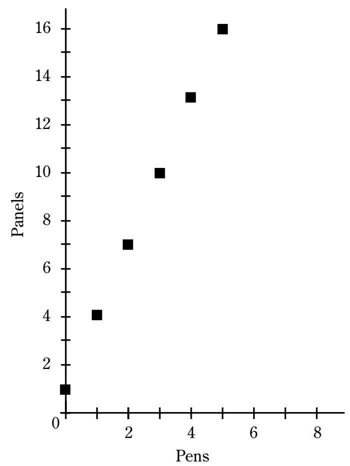


Fig. 3 Graphing points of a linear function

Investigating Whether Rules Are Equivalent

STUDENTS ARE OFTEN SURPRISED THAT MORE than one way may exist to express a rule. This situation is an opportunity to use graphs and algebra to enhance their learning. Graphing either of the two rules in **figure 2** produces points that fall on a line (see **fig. 3**). The line has a slope of 3 and a y-intercept of 1. As with all rules that produce points on a line, the slope is the value added to each successive term in the recursive rule (in this case, the three panels added repeatedly) and the y-intercept is the output produced when the input is 0. (In this case, the single vertical toothpick needed before three panels completes the first pen.) After investigating several such linear rules, students begin to recognize these connections, deepening their understanding of important algebraic ideas.

When two rules produce the same graph, we have evidence that they are, in fact, two forms of the same rule. To be perfectly sure, however, we need to use algebra to prove the equivalence. For example, using the rules in **figures 2a** and **2b,** we can show equivalence using the distributive property and adding like terms:

$$4 + 3(n - 1) = 4 + 3n - 3 = 1 + 3n$$

Unfortunately, not all patterns can be as easily derived as the case of the line illustrated above is. Sometimes, a geometric approach works better. Recall the earlier discussion that when shown the sequence of perfect squares, students often see only the recursive pattern, that is, next = current + next odd number. **Figure 4** offers an activity to help students recognize the equivalence of their recursive rule ("adding the next odd") and the explicit rule, n^2. Students are usually surprised and intrigued that they can physically see the successive odd numbers of a recursive rule being added to produce the squares of an explicit rule.

Investigating Patterns with Calculators

CALCULATORS CAN ALSO BE USED TO HELP STUDENTS explore linear and geometric sequences. One of the calculators that is helpful in exploring recursive sequences is the TI-73. With such a calculator, students can create recursive sequences using the constant feature. As shown in **figure 5a,**

Materials: Square color tiles

1. Start with 1 color tile. How many odd numbers have you used? What is the area of the square?

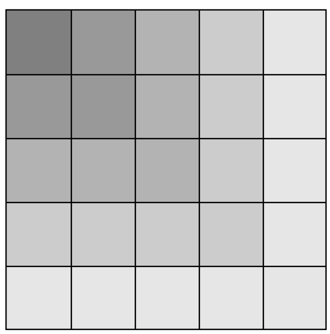
2. Add 3 more tiles of another color along the right and bottom edges to produce a square. How many odd numbers have you added so far? What is the area of the square?

3. Add 5 more tiles of another color along the right and bottom edges to produce a square. How many odd numbers have you added so far? What is the area of the square?

4. Continue adding 7, then 9 more tiles.

5. Describe your findings.

Partial response:



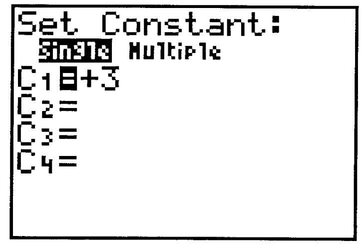
First number:	1 = 1
First two numbers:	1 + 3 = 4
First three numbers:	1 + 3 + 5 = 9
First four numbers:	1 + 3 + 5 + 7 = 16

The sums of consecutive odd numbers are perfect squares. You are adding enough to match each existing edge plus one more to fill in the corner of the new square.

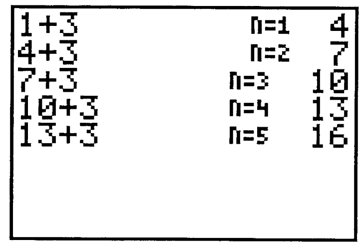
Fig. 4 The sum of consecutive odds equals a perfect square.

a student can SET an operation as a constant along with a value, for example, +3, which corresponds to our earlier example. Then, when the student presses 1 CONST, the calculator automatically prints and computes 1 + 3 and shows 4. The screen also displays n = 1, indicating the *first* use of the built-in constant. As the constant key is pressed repeatedly, successive terms in the sequence appear, along with their term numbers (n). **Figure 5b** shows the same procedure for producing a geometric sequence.

Students may like to verify that the same se-

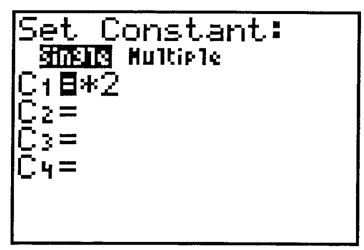


Press Set Constant, select Single mode, and set C1 = +3.

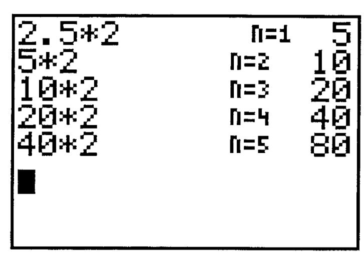


Press 1, then CONST (constant). The calculator automatically prints and calculates +3 and tells the value for n of the term. Successively pressing CONST produces successive terms in the sequence.

(a)
Linear sequence



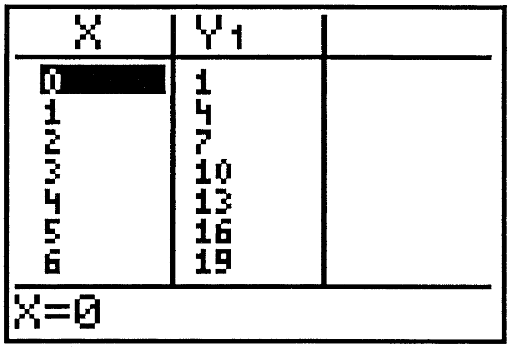
Press Set Constant, select Single mode, and set C1=*2.



Press 2.5, then CONST (constant). The calculator automatically prints and calculates *2 and tells the n of the term. Successively pressing CONST produces successive terms in the sequence.

(b)
Geometric sequence

Fig. 5 Using the TI-73 constant feature to produce linear and geometric sequences recursively



With Y= 3X + 1 in the Y= Menu

Fig. 6 Using the TI-73 table feature to produce sequences explicitly

quence is produced when using the explicit form of a rule. This time, they may use the Y= menu to enter a formula, as shown in **figure 6.** Using TBLSET, students can set the table to increment *x* values by ones. Finally, they can produce the table and compare the values produced explicitly with those produced recursively.

Summary

FAMILIARITY WITH NUMERICAL PATTERNS IS fundamental to students' number sense, problem solving, mental mathematics, modeling, and algebra concept learning. Being able to recognize, distinguish, and symbolize these patterns in both explicit and recursive forms is part of basic mathematics literacy. The function game and its input/output tables can be effective tools in helping students to achieve these goals.

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