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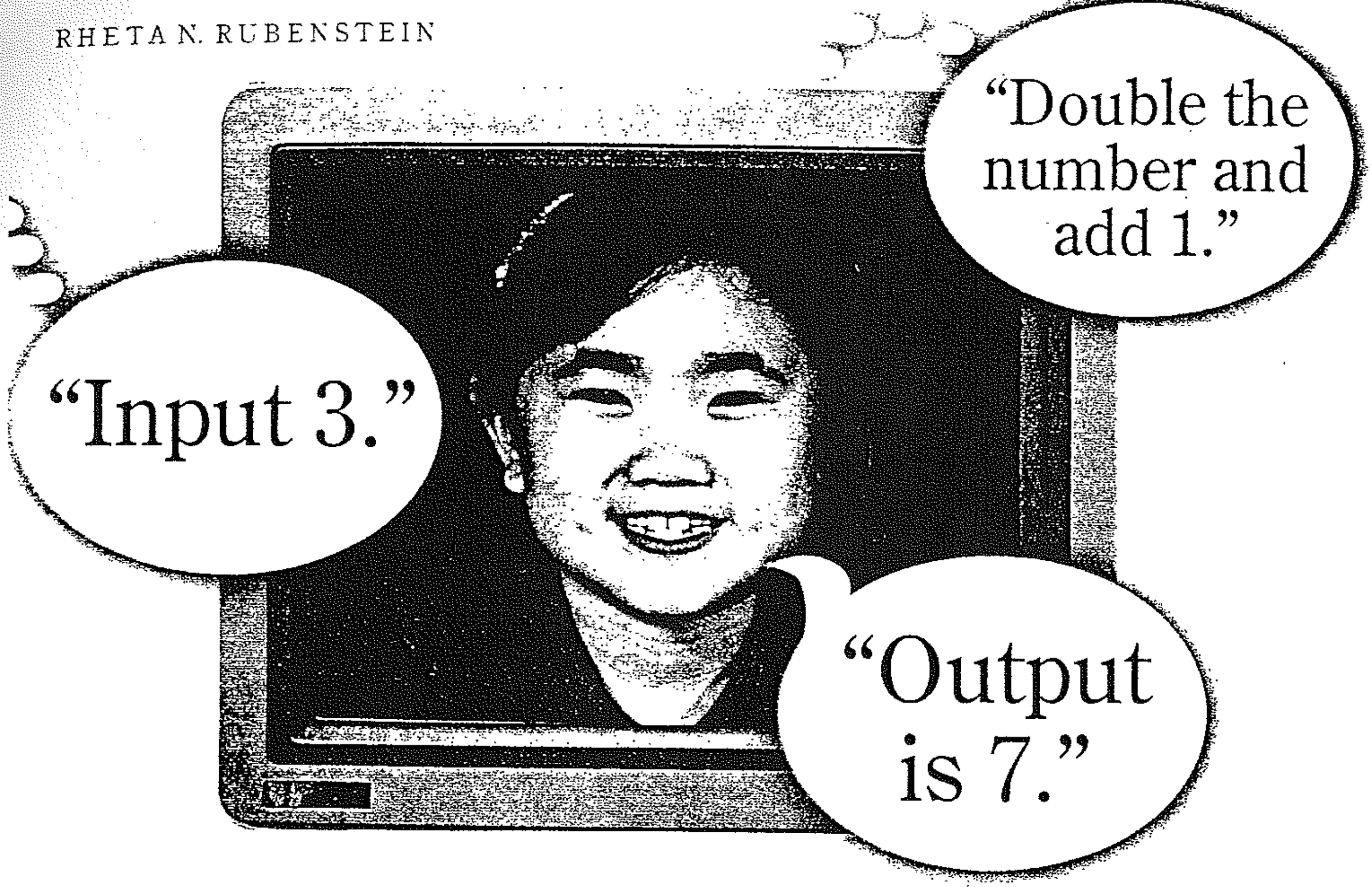

THE FUNCTION GAME

WHEN I BEGAN TEACHING IN THE late 1960s, we had no videotapes, commercial manipulatives, or calculators to create engaging learning activities for students. I recall spending a good part of my first year searching desperately for ways to motivate my seventh and eighth graders and help them learn mathematics. One activity that I stumbled across worked magically. I thought, "This is a gem," and I continue to cherish it today. I call it the "function game," but my students and others have called it the "input-output game," "guess my rule," or the "computer game." I understand that the game first gained prominence in the "new math" era, but it must have been around in some form much earlier. Today, the game is available on many computer systems and is popular with students and teachers. The computer versions, however, lack many dimensions of the live version.

I have used this game throughout my career with junior and senior high school students and, more recently, with preservice and in-service teachers. I find it to be flexible, engaging, and effective in teaching many essential mathematics concepts. Every time I use it, I continue to marvel at its power and potential. I will demonstrate how it works and share ideas for using it to teach lots of mathematics.

The rules are simple. One person is the "computer" and thinks of a rule, for example, "double the number and add 1." Classmates give "input" values, and the "computer" gives the corresponding output for each. These output values are recorded one at a time on a table as shown in **figures 1–9.** The object of the game is to guess the rule. When playing with a whole class, we do not want to spoil the fun quickly, so I tell the students that if they know the rule, they should not say it aloud but just tell the "computer" that they know it. Then the "computer" gives an input value and the player must give the output. If the answer is correct, it will be recorded in the table and play continues. When it becomes clear that most students know the rule, one player is allowed to state it. Another student is then chosen to be the next "computer."

Benefits of the Game

THE GAME HAS MANY BENEFITS. BECAUSE IT IS PREdicated on a "secret rule," students are intrigued and motivated to play. Because it is simple and requires no materials, it can be played at any time, for example, as a warm-up or brief closing activity or as part of the development of a lesson. More important, the game uses lots of mathematics. Even to determine simple rules, students must use mental mathematics. Clearly, it requires problem solving that can be very challenging, as shown subsequently. Playing also promotes many opportunities for communication. Moreover, as its name suggests, it is a function game and develops key ideas of algebra: variable, expression, function, and modeling. Let us see how some of these concepts are fostered.

Figures 1–9 show several games in action. Before reading on, try to guess the rules.

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Discourse about Equivalent Expressions

ONE OF THE MAJOR RECOMMENDATIONS OF THE NCTM's *Professional Standards for Teaching Mathematics* (1991) is for teachers to promote discourse in mathematics classes. This game definitely affords opportunities for class discussions. For example, when I use the rule in **figure 1,** I ask students to state it in as many different ways as they can. Some of their responses follow:

- "Double the number and take away two."
- "Take twice the number, then minus two."
- "Multiply by two, subtract two."

This activity gives us an opportunity to recognize that mathematical operations can be expressed in many equivalent ways, which leads to one of the major benefits of the game: the natural and meaningful introduction of variables. Among the multiple expressions of this rule that students may suggest—or can be shown—is $2n - 2$, where n represents any number. This expression illustrates the efficiency of using symbols and, when coupled with the earlier discussion of different expressions, shows how the language of algebra can generalize arithmetic.

Students are intrigued by the secret rule

Another expression that sometimes surfaces for the data in **figure 1** is "take one less than the number, then double it." If this answer is not suggested by a student, I will say, "I once had someone tell it to me like this. . . ." As with the other rules, we discuss it to see if it works. Depending on where students are in their work with variables, this discussion can serve as an excellent entrée to explore the equivalence of $2n - 2$ and $(n - 1)2$ or $2(n - 1)$. **Figure 2** shows another example in which two seemingly different rules work: multiply the number by the next number or square the number and add it to itself, which algebraically is $n(n + 1)$, or $n^2 + n$. These and similar examples are opportunities for introducing

Input	Output
7	12
12	22
4	6
8	14
3	4

Fig. 1

Input	Output
5	30
8	72
11	132
0	0
2	6

Fig. 2

Input	Output
2	8
5	5
12	–2
6	4
20	–10

Fig. 3

Input	Output
7	0
12	1
4	0
8	0
3	1
9	1

Fig. 4

Input	Output
5	0
8	3
11	1
0	0
2	2
9	4

Fig. 5

the concept of the distributive property in rewriting algebraic expressions.

Figure 3 is also interesting. Some students see it as "what you need to add to get ten," whereas others see it as "ten minus the number." I call it "the tens complement." Each new type of rule enlarges the students' repertoire of ideas with which to challenge their classmates in future games.

Number Theory and Number Sense

NUMBER THEORY, AN IMPORTANT TOPIC IN middle school mathematics, can be developed nicely through this game. For example, **figure 4** shows a rule that sorts multiples of 3; it outputs a 1 to say, "Yes, this input is a multiple of three" and a 0 to say, "No, this is not a multiple of three." Obviously this rule can be extended to other multiples. **Figure 5** is usually a baffling rule initially. Then, as they do with many earlier games, students get wise and decide to give inputs in sequence to see better what is happening. Often they will see the pattern but may not be able to explain it, saying, "It goes zero, one, two, three, four, zero, one, two, three, four, and so on." Sometimes it helps to ask them to think about the numbers that produce zero. Once they see that the rule can also be expressed as the remainder when the input is divided by 5, their repertoire is again expanded and they have been introduced to modular arithmetic.

The rule in **figure 6** can be tough unless students have recently been working with prime numbers. If they have, and if they have seen other "sorting machines" as in **figure 4,** then someone usually figures out that it is a prime sorting machine, producing a 1 for primes and a 0 for nonprimes. Often students need more data than the set shown. Of course, once this machine is introduced, it can be reused later simply to provide practice in recognizing primes.

Figure 7 illustrates a rule that challenges students' number sense. For each input, the "computer" produces the whole-number part of the square root of the number. This rule combines the concept of square root and what is sometimes called the "floor function" or the "round down function." Related rules that can also be used are "round to the nearest" or "round up," which is sometimes called the "ceiling function."

The function game can also introduce notions of domain and range, the acceptable input values of a function and the resulting outputs, respectively. For example, I have told students, "The next machine takes decimal values" or "This one accepts fractions." When they have done this activity a few times, I introduce the term *domain* as the set of acceptable inputs. Then I can say such things as "The domain of this function is improper fractions." In addition to introducing new vocabulary, these variants allow us to use more rules. For example, if the domain is decimal numbers, good rules to use are rounding, multiplying or dividing by 10s or 100s, adding 20, and so on. If the domain is improper fractions, good rules are renaming as a mixed number, rounding to the nearest whole number, adding 1, or doubling. In each instance, we continue to talk about multiple ways to express the rule. In this way, for example, students can see that multiplying a decimal by 10 and "moving the decimal point one place to the right" are equivalent. Whereas some of these rules might not challenge students for long in the "guess my rule" aspect of the game, they do offer engaging practice and mental mathematics.

Probably the most challenging rule I ever invented for my eighth graders was the one shown in **figure 8.** They had played the game many times and were pretty good at it, but this rule kept them going for a long, long time. After we had filled three or four tables of values and no one had found the rule, I said that if anyone got it, he or she could simply state it. Interest was very high. Finally, from the back corner of the room a young man volunteered who rarely participated and whom the others perceived as being a nonparticipant. "It's the next prime number," he announced. Students saw immediately that his answer was correct. The class was stunned. An audible silence was followed by gasps, then by congratulations. I believe that everyone's perception of the student was profoundly changed in this instant. This experience was definitely a telling moment in my career. It confirmed dramatically my belief that we never really know the depth of our students.

Simplicity Made Complex

THE RULE IN FIGURE 9 ALSO HAS AN INTERESTING, BUT more recent, story. I was doing a workshop with pre-service elementary teachers. I had introduced the game, and we had played several rounds together. Then I asked them to take turns playing at their tables. After a while, I asked each group to choose what it thought was the most interesting rule and to play it with the whole class. One student's rule produced a table like the one in **figure 9.** I loved it. I had never before dreamed of playing a constant rule, and I thought that this idea was simple, yet important. I was even more impressed to see the student teacher take long pauses after each input while she seemed to be calculating the output. "What drama!" I thought, "She's really playing it up!" The class, too, was engrossed. At one point, a player asked for the output for pi. The "computer" said she would have to round it, wrote 3.14, and paused for a really long time, then wrote 4 again.

After different students had said that they knew the rule and had given correct outputs, I felt that it was time to move on and asked for someone to state the rule, which in my mind was simply "write four." A student said, "Subtract the number from itself and add four." I was surprised. As enamored as I was of the notion of multiple rules, I really had not thought about this one as a calculation. Other students objected. One said, "No, my rule was 'Multiply by zero then add four.' " Then the "computer" said, "No, my rule was 'Divide the number by itself and multiply by four.' " I realized then that her long pauses were not drama; she was really calculating for each input! Rumblings continued. We began recording the various rules. Many were like "double the number, divide by the number, then add two" or "add seven, subtract the number, then subtract three." I thought that surely, if I waited long enough, someone would state my simple version. No one did. Finally, I said, "I thought about it another way. Whatever the input is, the output is four." Frowns appeared. They did not like this answer at all. People thought that I was cheating! "You're supposed to 'do' something with the input," they told me. What I learned was that in their perception of a "rule," "steps" or "procedures" must occur. In their estimation, arithmetic, not insights, yielded the only legal moves. I realized that the notion of a constant rule, like many other concepts in mathematics, is so simple that it is challenging.

In retrospect, I regretted that I had missed a golden opportunity. Their notions of working with zeros and ones and with steps that undid themselves had offered an opportunity to help them learn about identities and inverses. Alas, not every teachable moment is captured!

Extensions and Variants

ANOTHER STRENGTH OF THE GAME IS THE possibility of extensions and variants. One major extension is to have students graph the ordered pairs. Rules like $n + 2$, $n + 5$, and $n - 3$ can be graphed and compared. They produce points on parallel lines. Rules like $2n + 1$, $3n + 1$, $(1/2)n + 1$, and $-n + 1$ produce points on lines that all intersect the y-axis at 1. Rules like $n(n + 1)$ or n^2 or $n(n - 2)$ produce points on a curve, specifically, a parabola. Such functions grow more quickly than do those that produce lines. They afford excellent opportunities to introduce and contrast linear and quadratic growth, although not necessarily in those terms, and, in the case of lines, to introduce concepts like slope and intercept. Also, once students realize that graphs can help them see the rules, graphs, too, become another tool for playing the game.

As another extension, students can be asked questions about what is or is not possible. For example, after playing the rule "double the number and add 1," ask students, "If the domain is the set of integers, can the computer produce twelve?" Reasoning questions like this one pave the way for later work with proof.

Another variant of the game is to play in reverse. Say, "If you know the rule, I'll give you the output and you give me the input." In other words, the students are inverting the rule. Only rules that have unique inputs for each output will work, however. For example, any linear-function rule will work, but not rules involving squares or rounding. This variant is an excellent way to get students started with equation solving. For example, if the rule is $2n + 1$ and the output is 13, to find the input they must subtract the 1 then divide by 2 to get 6. **Figure 10** illustrates the process with a diagram. This variant is a chance to talk about how we undo expressions in the opposite order in which we

Input	Output
2	1
5	1
12	0
6	0
20	0
11	1

Fig. 6

Input	Output
2	1
5	2
12	3
6	2
20	4
36	6

Fig. 7

Input	Output
7	11
12	13
4	5
8	11
3	5
9	11

Fig. 8

Input	Output
5	4
8	4
11	4
0	4
2	4
9	4

Fig. 9

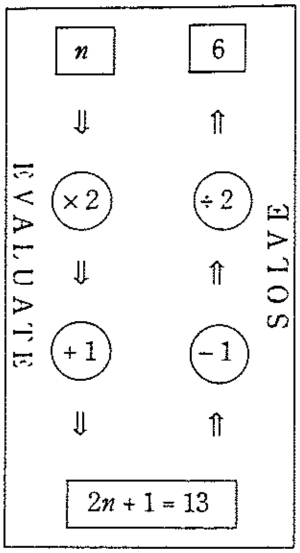


Fig. 10

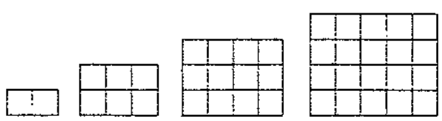
evaluate them and how we use inverse operations at each step.

The game can also be played with two inputs. For example, with rational numbers as the domain, rules could be "pick the larger value," "add," or "average the two values."

The game can also be connected to geometry with diagrams and materials. Both the elementary- and middle-grades NCTM Addenda books on patterns, *Patterns* (Coburn et al. 1993) and *Patterns and Functions* (Phillips et al. 1991), respectively, contain several examples of geometric rules. **Figure 11** shows activities adapted from the grades K–6 Addenda book (Coburn et al. 1993). Item 1 in **figure 11** is $n(n + 1)$, which was mentioned earlier. Item 2 illustrates "double the previous term number and add 1" or "the sum of the term number and the previous term number." This item gives another chance to show equivalence, this time of $2(n - 1) + 1$ and $n + (n - 1)$. All students benefit from visualizing algebraic expressions geometrically; this experience may be particularly important for students who learn better visually. The grades 5–8 Addenda book (Phillips et al. 1991) presents other excellent examples of geometric-patterning activities. Anno (1989), in his "Magic Machine" chapter, also offers appealing nonnumerical visual images of the function idea that can be used with young as well as middle-grades children.

The game can also help build an essential idea for probability: random numbers. This version requires a calculator. Most calculator random-number generators produce a random number between 0 and 1. One way to play this "rule" for any input is to give the first digit in the decimal display of a random number. Other ways would be to give the first two digits, and so on. In any example, the "pattern" that students need to recognize is the absence of a pattern. The output is unpredictable. This variant illustrates another simple, yet important, mathematical idea.

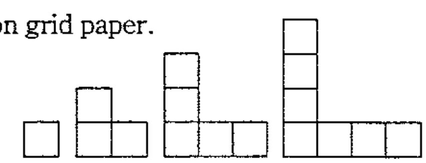
1. Find the number of unit squares in the rectangles



Term	1	2	3	4
Squares	2	6	12	?

a) How many unit squares in the 8th term?
b) How many unit squares in the *n*th term?
c) Is 81 a possible number of squares in some term in the pattern? How do you know?
d) If you know the number of unit squares in the *n*th term, how can you find the number of squares in the next term?

2. Make Ls on grid paper.



Term	1	2	3	4
Squares	1	3	5	7

a) Extend the pattern. Find expressions for the *n*th term.
b) Make a new pattern: Combine the first two figures, then the first three, then the first four, and so on. What new shapes do you get? How many squares? Describe the pattern.

Fig. 11 Pattern and activities with grid paper (adapted from Coburn et al. [1993])

Conclusion

THE FUNCTION GAME OFFERS MANY OPPORTUNITIES to introduce, develop, practice, extend, communicate about, and connect mathematical ideas. Students of all ages enjoy playing and learning with it. Readers are invited to try the function game with their students and explore its remarkable potential.

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